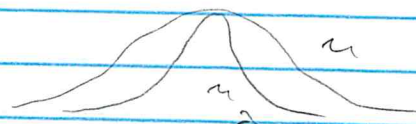


HM seminar talk, 10/24/18 @ MSU.

~1:40.

$$u: \mathbb{R}^m \rightarrow \mathbb{N}, \text{ cpt.}$$

$$u_\lambda(x) = u(x/\lambda)$$



$$\nabla u_\lambda = \lambda^{-1} \nabla u(x/\lambda).$$

$$\int_{B_\lambda} |\nabla u_\lambda|^2 dV = \lambda^{-2} \int_{B_1} |\nabla u|^2(x/\lambda) dV.$$

$$\lambda^{2-m} \int_{B_{\lambda(x_0)}} |\nabla u_\lambda|^2 = \int_{B_1} |\nabla u|^2.$$

||

$$\Phi_{\lambda, x_0}(u).$$

central oscillates on smaller balls

Lemma (Morrey): $\|u\|_{C^d} \leq C \left\{ \sup_{0 < \lambda \leq \lambda_0} \lambda^{-2d} \Phi_{\lambda, x_0}(u) \right\}$

If u is harmonic, $u \in C^d \Rightarrow u \in C^{1,2} \Rightarrow u \in C^\infty$.
 (note: w.p. doesn't work. $\Delta u \Rightarrow A(u)(\nabla u, \nabla u) \in L^1$...)

Prop: Let u be a harmonic map

satisfying $\Phi_{\lambda, x_0}(u) \leq C \lambda^{2d}$ for some $\lambda > 0$

$$\forall x_0 \in M, 0 < \lambda \leq \lambda_0.$$

Then u is C^∞ .

○ Theorem (Morrey). If $n=2$, then any minimizing u is regular.

Pf: $n=2 \Rightarrow \Phi_\lambda(u) = \int_{B_\lambda} |\nabla u|^2 dV$.

Need to show $\int_{B_\lambda} |\nabla u|^2 \leq C \lambda^{2\alpha}$, same $\alpha > 0$.

Let $\delta > 0$ s.t. $\pi: N_\delta \rightarrow N$ is C^α .

For any $x_0 \in M$ and $\varepsilon > 0$, $\exists \lambda_0 > 0$ s.t.

$$\int_{B_{\lambda_0/2}(x_0)} |\nabla u|^2 < \varepsilon$$

~~Rescale wlog s.t. $\lambda_0 = 1$.~~

Claim: $\forall \lambda < \lambda_0$, we have

$$\int_{B_{\lambda/2}} |\nabla u|^2 \leq C \int_{B_\lambda \setminus B_{\lambda/2}} |\nabla u|^2$$

Pf: Wlog rescale s.t. $\lambda = 1$.

$$\Rightarrow \int_{B_1} |\nabla u|^2 < \varepsilon^2.$$

Fubini

$\Rightarrow \exists r_0 \in (1/2, 1)$ s.t.

$$\int_{S_{r_0}} |\nabla u|^2 r_0 < 2 \int_{B_1 \setminus B_{r_0/2}} |\nabla u|^2 < \varepsilon^2.$$

$$\Rightarrow \int_{S_{r_0}} |\nabla u|^2 < 2\varepsilon^2.$$

1-0 Posinging: $\text{osc } u \in C_c$ $\frac{1}{s_{r_0}}$

$$\Rightarrow u: s'_{r_0} \rightarrow B_{c_c} \subset N_S \text{ if } u \in \frac{1}{s}$$

$$\text{Let } v: B_{r_0} \rightarrow B_{c_c} \subset N_S \subset \mathbb{R}^N$$

$$\text{set } \Delta v = 0$$

$$v|_{s'_{r_0}} = u|_{s'_{r_0}}$$

$$\text{Max. prin} \Rightarrow \text{Im}(v) \subset B_{c_c}$$

$$\int_{B_{r_0}} |\nabla u|^2 \leq \int_{B_{r_0}} |\nabla(\pi v)|^2$$

$$\pi_S \text{ is } C^\infty \quad \leq C \int_{B_{r_0}} |\nabla v|^2$$

~~$$\text{Calc. m.v.p.} \leq C \int_{s'_{r_0}} |\nabla u|^2$$~~

$$u = v \text{ on } s'_{r_0} \leq C \int_{s'_{r_0}} |\nabla u|^2$$

$$\int_{B_{1/2}} |\nabla u|^2 \leq C \int_{B_1 \setminus B_{1/2}} |\nabla u|^2$$

~~$$\text{But } \int_{B_{1/2}} |\nabla u|^2 = \int_{\Omega} -$$~~

But $\int_{B_{1/2}} |Du|^2 \leq \int_{B_1} |Du|^2 - \int_{B_1 \setminus B_{1/2}} |Du|^2$

$$\leq \int_{B_1} |Du|^2 - \frac{1}{C} \int_{B_{1/2}} |Du|^2$$

$$\left(1 + \frac{1}{C}\right) \int_{B_{1/2}} |Du|^2 \leq \int_{B_1} |Du|^2$$

Let $\delta = \frac{1}{2} \left(\log_2 \left(1 + \frac{1}{C} \right) \right)$

□

Crucial facts: could assume $\int_{B_1} |Du|^2 < \varepsilon^2$
and S^1 maps into a ball.

Theorem: (Liu 04.3) Assume n is a
(non-convex).
weakly harmonic map whose image is cent'd on
a "regular" ball $B(y_0) \subset N$ (i.e. $\text{dist}_N^2(y_0, \cdot)$ is
convex on $B(y_0)$)
Then u is C^∞ .

Counter: $n \geq 3$, $u = \frac{x}{|x|} : \mathbb{R}^n \rightarrow S^{n-1}$
is a minimizing Hdr! (notice image not
cent'd on ball)
Also: can't approx!

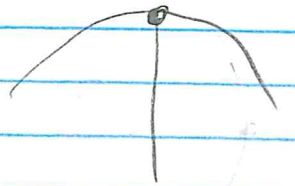
Pf. (Lm) Let $\phi: \mathbb{R}^2 \rightarrow S^{n-1}$

arb. map w/ $\phi|_{S^{n-1}} = Id.$

Claim: $|\nabla \phi|^2 \geq \frac{1}{n-2} ((\operatorname{div} \phi)^2 - \operatorname{tr} |\nabla \phi|^2)$ (a.e.)

Assume $\phi(x_0) = (0, \dots, 0, 1)$

Let $a_{ij} = \partial_i \phi^j(x_0)$ *Careful with indices!*



$$\Rightarrow a_{ij} = 0 \quad \forall j.$$

$$(\operatorname{div} \phi)^2(x_0) = \left(\sum_{1 \leq i \leq n-1} a_{ii} \right)^2 \leq (n-1) \sum a_{ii}^2$$

$$|\operatorname{tr} \phi|^2 = a_{ii} a_{jj} = \sum a_{ii}^2 + \sum_{i \neq j} a_{ii} a_{jj}$$

$$\text{Cauchy-Schwarz} \quad \geq \sum a_{ii}^2 - \sum_{i \neq j} a_{ii}^2$$

$$\Rightarrow (\operatorname{div} \phi)^2 - \operatorname{tr} |\nabla \phi|^2 \leq (n-2) \sum a_{ii}^2 + \sum_{i \neq j} a_{ii}^2 \leq (n-2) |\nabla \phi|^2 \quad \text{Helm.}$$

But actually:

$$\begin{aligned} (\operatorname{div} \phi)^2 - \operatorname{tr} |\nabla \phi|^2 &= \partial_j \phi_i \partial_i \phi_j - \partial_i \phi_j \partial_j \phi_i \\ &\quad + \partial_i \partial_j \phi_j \phi_i - \partial_i \partial_j \phi_i \phi_j \\ &= \partial_i (\partial_j \phi_j \phi_i - \partial_j \phi_i \phi_j) \end{aligned}$$

$$\Rightarrow \int_{B_1} |\nabla \phi|^2 \geq \frac{1}{n-2} \int_{B_1} ((\Delta \phi)^2 - n |\nabla \phi|^2)$$

$$= \frac{1}{n-2} \int_{\partial B_1} (\partial_j \phi \cdot \phi_i - \partial_j \phi_i \cdot \phi_j) x^i$$

$$\phi = x^i$$

$$= \frac{1}{n-2} \int_{S_1} (\partial_j \phi_j - x^j \partial_j \phi^i x^i)$$

(a_{jj} - a_{nn} = 0)

$$= \frac{1}{n-2} \int_{S_1} (n-1)$$

all perpendicular
vectors = $\frac{x}{|x|}$

$$= \frac{n-1}{n-2} |S^{n-1}| = \int_{B_1} |\nabla \frac{x}{|x|}|^2 \quad \square$$

But $\nabla \frac{x}{|x|} = \frac{S^{n-1}}{|x|}$ (all v are = e_n)

$$\Rightarrow |\nabla \frac{x}{|x|}|^2 = \frac{n-1}{|x|^2}$$

$$\int |\nabla \frac{x}{|x|}|^2 = \frac{n-1}{n-2} \int_{B_1} \frac{1}{|x|^2} = \frac{n-1}{n-2} |S^{n-1}|$$

Good news: for minimizing
(or stationary) Hdr's,

$$\begin{aligned} \Phi_{r,x_0}(u) &\leq \Phi_{t,x_0}(u) \quad \text{if } r \leq t \\ &\Leftrightarrow \frac{\partial \Phi}{\partial r} = 0. \end{aligned}$$

Theorem (Golau-Whl.)

$\Rightarrow \epsilon_0 = \epsilon_0(u, N) > 0$ s.t. if
 $u \in B_{\epsilon_0} \rightarrow N$ is a min h.m. with

$$\Phi_1(u) = E(u, B_1) = \epsilon^2 \leq \epsilon_0^2$$

Then $u \in C^\infty(B_{r_2})$.

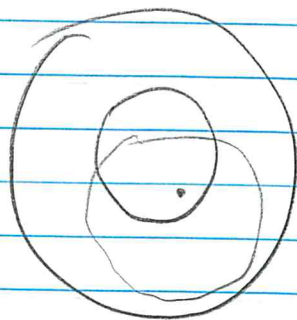
~~Proof~~ ^{sketch} By Morrey's lemma, it suffices
to show

$$\Phi_{\theta_0}(u) \leq \frac{1}{2} \Phi_1(u)$$

for some $0 < \theta_0 < 1$.

Notice that for all $x \in B_{1/2}$,
by monotonicity, we have

$$\begin{aligned} |\bar{\Phi}_{r,x_0}(u)| &\leq |\bar{\Phi}_{1/2,x_0}(u)| \\ &\leq C \cdot \bar{\Phi}_{1,0}(u) \\ &\leq C \varepsilon^2. \end{aligned}$$



Idea: use mollifier trick to get

Image cent. on a ball. Let $\phi(y)$ smooth,

$0 \leq \phi \leq 1$, let $\phi^h(y) = h^{-n} \phi(y/h)$. $\int \phi = 1$, $\text{supp } \phi \subset B_1$.

$$u^h(x) = \int_{B_1} \phi^h(x-y) u(y) dy$$

Then, for $x \in B_{1/2}$, have

$$\begin{aligned} \text{dist}^2(u^h, N) &\leq \frac{1}{|B_n(x)|} \int_{B_n(x)} |u(y) - u^h(x)|^2 dy \\ &\leq C \cdot h^{2-n} \int_{B_n} |\nabla u|^2 dy \leq C \varepsilon^2. \end{aligned}$$

$\Rightarrow u^h \in N_\delta!$

Moreover, if we let $\bar{h} = \varepsilon^{1/2}$, then

$$|\nabla u^{\bar{h}}|^2(x) = \left| \int_{B_1} \phi^{\bar{h}}(x-y) \Delta u(y) dy \right|^2$$

$$\leq \int_{B_1} \phi^{\bar{h}}(x-y) |\nabla u|^2(y) dy$$

$$\leq C \cdot \bar{h}^{-n} E_{B_{\bar{h}}}(x)(u)$$

$$\leq \bar{h}^{-2} \varepsilon^2 \leq C \varepsilon$$

$$\Rightarrow \left[\sup_{x \in B_{1/2}} |\nabla u^{\bar{h}}| \leq C \sqrt{\varepsilon} \right]$$

Now, choose $r_0 \in [1/2, 1/2]$ s.t.

$$\int_{S_{r_0}^{n-1}} |\nabla u^{\bar{h}}|^2 \leq C \varepsilon^n$$

Let $\Delta v = 0$ on B_{r_0} ,

$$v = u \cdot \bar{h} \text{ on } S_{r_0}^{n-1}$$

Then $\sup |v - u \bar{h}| \leq 2 \sup u \bar{h} \leq C \varepsilon^{1/2}$

and by std. est,

$$\sup_{B_{1/4}} |\nabla v|^2 \leq C \int_{S_{r_0}^{n-1}} |\nabla u|^2 \leq C \varepsilon^n$$

Hence, for $\theta \leq 1/4$,

$$\begin{aligned} \Phi_{\theta,0}(u^{\bar{u}}) &\leq 2\theta^{2-\gamma} \left(\int_{B_\theta} |\nabla u^{\bar{u}} - \nabla v|^2 \right. \\ &\quad \left. + \int_{B_\theta} |\nabla v|^2 \right) \\ &\leq 2\theta^{2+\gamma} \int_{B_{r_0}} |\nabla(u^{\bar{u}} - v)|^2 + C\theta^2 \varepsilon^2. \end{aligned}$$

Notice: $\int_{B_{r_0}} |\nabla(u^{\bar{u}} - v)|^2$ (no bad term
 b/c $u = v$ on ∂B_{r_0})

$$= - \int_{B_{r_0}} \Delta u^{\bar{u}} (u^{\bar{u}} - v)$$

$$\begin{aligned} \text{but } \Delta u^{\bar{u}} &= \int_{\mathbb{R}^n} \Delta_x \phi^{\bar{u}}(x-y) u(y) dy \\ &= \int_{\mathbb{R}^n} \Delta_y \phi^{\bar{u}}(x-y) u(y) dy \end{aligned}$$

Wah Han!

$$= \int_{\mathbb{R}^n} \phi^{\bar{u}}(x-y) A(u)(\nabla u, \nabla u) dy$$

$$\Rightarrow \int_{B_{r_0}} |\Delta u^{\bar{u}}| \leq C A(u, \nabla u) |\nabla u|^2$$

$$\int_{B_{r_0}} |\nabla(u^{\bar{u}} - v)|^2 \leq$$

$$\Rightarrow \left| \int_{B_{r_0}} \Delta u^{\bar{u}} (u^{\bar{u}} - v) \right| \leq C \sqrt{\varepsilon} \int_{B_{r_0}} |\nabla u|^2 \leq C \sqrt{\varepsilon} \varepsilon(u, \nabla u)$$

1)

$$\Rightarrow \Phi_{\theta_0}(u) \leq C(\theta^{2-\eta} \varepsilon^{1/2} + \theta^2) E(u, B)$$

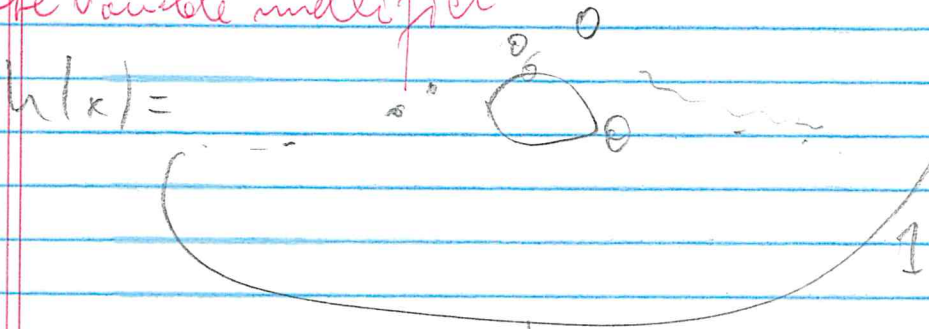
$$\text{choose } C\theta^2 < 1/4$$

$$\text{then } \varepsilon \text{ s.t. } C\theta^{2-\eta} \varepsilon^{1/2} < 1/4$$

$$\Rightarrow \Phi_{\theta_0}(u) \leq \frac{1}{2} \varepsilon^2$$

$\forall$ for u^h

Use variable multiplier



~~Use comparison~~ ~~bound on u~~

$$u(x) = \inf u$$

Comparison $\rightarrow$ bound on u !

Do Jensen's est.